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# Generalized exclusion and Hopf algebras

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Received 19 July 2002, in final form 23 September 2002

Published 4 December 2002

Online at [stacks.iop.org/JPhysA/35/10803](http://stacks.iop.org/JPhysA/35/10803)

## Abstract

We propose a generalized oscillator algebra at the roots of unity with generalized exclusion and we investigate the braided Hopf structure. We find that there are two solutions: these are the generalized exclusions of the bosonic and fermionic types. We also discuss the covariance properties of these oscillators.

PACS numbers: 02.10.Hh, 02.20.Uw, 03.65.Fd

## 1. Introduction

In principle, the generalization of the statistics of identical particles can be performed in two different ways. The first approach is based on the generalization of the  $\pm 1$  factor of boson/fermion algebra, which comes from the interchange of two identical particles. This approach leads to the introduction of anyons [1] which have found applications in the fractional quantum Hall effect and superconductivity [2]. Quon algebras [3] and braided oscillators [4] are other generalizations based on the generalizations of the exchange parameter. Mathematically, this corresponds to the generalization of the algebra in tensor product space such that

$$(1 \otimes a)(b \otimes 1) = \pm \pi(a \otimes b) = \pm b \otimes a \quad (1)$$

where the plus and minus signs refer to bosons and fermions, respectively. The permutation map  $\pm \pi$  is replaced by a generalized map  $\psi$  i.e.

$$(1 \otimes a)(b \otimes 1) = \psi(a \otimes b). \quad (2)$$

In general, we have  $\psi^2 \neq id$  and hence the map  $\psi$  is called braiding. The algebra in the tensor product space is connected with Hopf algebras such that the coproduct map  $\Delta$  of Hopf algebras is a homomorphism to the tensor product space  $\Delta : A \rightarrow A \otimes A$ . It creates identical copies of algebras and hence the theory of Hopf algebras can be used in the operator algebra formulations of identical particles. For bosons/fermions ordinary/super Hopf algebras are used, and axioms are used for the generalized exchange phase braided Hopf algebra [5]. The

braided Hopf algebra axioms are generalizations of ordinary Hopf algebra axioms such that, in the limit  $\psi \rightarrow \pi$ , braided Hopf algebra axioms reduce to the ordinary Hopf algebra axioms.

Another approach for the generalization of statistics is based on the generalization of the Pauli exclusion principle proposed by Haldane [6] and the statistical mechanics is developed by Wu [7]. In the literature, an extensive amount of work has been carried out as applications of this generalization. Our aim in this paper is to find an operator algebra for this generalized exclusion. It is known that the  $q$ -deformed generalization of nilpotent algebras requires that the deformation parameter is a root of unity [12]. For example, the  $q$ -deformed harmonic oscillators at the roots of unity is connected with two anyon system [8]. Different versions of the deformation of oscillator algebras and their use in physics can be found in the review [9] and the connection of between  $q$ -deformed structures and fractional statistics can be found in [10]. In terms of creation and annihilation operators, the generalized exclusion can be expressed as  $a^k = 0$  and  $(a^\dagger)^k = 0$ . The limit  $k \rightarrow 2$  gives the Pauli exclusion principle for fermions. Some operator algebras for fractional exclusion statistics are proposed in [11].

In this paper, we discuss the problem of the construction of an oscillator algebra unifying the notions of generalized phase (braiding), Hopf algebra and exclusion statistics, i.e. we investigate the braided Hopf algebra structure of a generalized oscillator with generalized exclusion  $a^k = 0$ .

## 2. Generalized exclusion algebra

We start with a generalized oscillator algebra generated by  $(a, a^*, q^N, q^{-N}, 1)$  satisfying

$$\begin{aligned} aa^* - Q_1 a^* a &= Q_2 q^{2N} + Q_3 q^{-2N} & aa^* - Q_1^{-1} a^* a &= Q_2^* q^{-2N} + Q_3^* q^{2N} \\ aq^N &= qa^N a & q^N a^* &= qa^* q^N \\ aq^{-N} &= q^{-1} q^{-N} a & q^{-N} a^* &= q^{-1} a^* q^{-N} \\ q^N q^{-N} &= q^{-N} q^N = 1_A & a^k &= (a^*)^k = 0 \end{aligned} \quad (3)$$

where  $q$  and  $Q_1$  are complex parameters of unit modulus,  $Q_2$  and  $Q_3$  are any complex parameters, and  $1_A$  is the identity of the algebra. For the  $*$ -structure we impose

$$(a^*)^* = a \quad (q^{\pm N})^* = q^{\mp N}. \quad (4)$$

From the relations (3) we obtain

$$\begin{aligned} a^* a &= \frac{(Q_2 - Q_3^*) q^{2N} + (Q_3 - Q_2^*) q^{-2N}}{Q_1^{-1} - Q_1} \\ aa^* &= \frac{(Q_1^{-1} Q_2 - Q_1 Q_3^*) q^{2N} + (Q_1^{-1} Q_3 - Q_1 Q_2^*) q^{-2N}}{Q_1^{-1} - Q_1}. \end{aligned} \quad (5)$$

We note that, in contrast to the operator algebra proposed by Karabali–Nair [11] where the number operator is a function of the raising and lowering operators, i.e.  $N = f(a^* a)$ , we take the Hermitian operator  $a^* a$  as a function of the number operator, i.e.  $a^* a = f(N)$ . Since  $a$  and  $a^*$  are lowering and raising operators, respectively, the operator  $aa^*$  should satisfy  $aa^* = f(N + 1)$ . The function  $f(N)$  for the algebra we propose is given by (5).

$$q^2(Q_2 - Q_3^*) = Q_1^{-1} Q_2 - Q_1 Q_3^*. \quad (6)$$

The special cases  $Q_1 = q^{-2}$  and  $Q_1 = q^2$ , with a rescaling of the generators, reduce to

$$aa^* - q^{-2} a^* a = q^{2N} \quad aa^* - q^2 a^* a = q^{-2N} \quad (7)$$

which is the well-known  $q$ -oscillator algebra [13]. At the roots of unity, this algebra is related to two anyon system [8]. For  $Q_1 \neq q^2, q^{-2}$  we express  $Q_3$  in terms of  $Q_2$ . Then the oscillator algebra turns out to be

$$\begin{aligned} aa^* - Q_1 a^* a &= Q_2 q^{2N} + Q_1 R Q_2^* q^{-2N} \\ aq^N &= qa^N a \quad q^N a^* = qa^* q^N \quad a^k = 0 \end{aligned} \tag{8}$$

and their  $*$ -conjugates. The parameter  $R \equiv \frac{1-q^2 Q_1}{Q_1 - q^2}$  is real. For the Fock space representation we take

$$a|n\rangle = a_n|n-1\rangle \quad a^*|n\rangle = a_{n+1}^*|n+1\rangle \quad q^N|n\rangle = q^n|n\rangle. \tag{9}$$

Solving the difference equations, we obtain the spectrum

$$a^* a|n\rangle = \frac{Q_2 q^{2n} - Q_2^* Q_1 q^{-2(n-1)}}{q^2 - Q_1} |n\rangle. \tag{10}$$

The existence of a ground state  $a|0\rangle = 0$  implies

$$Q_2^* = q^{-2} Q_1^{-1} Q_2. \tag{11}$$

Since  $q$  and  $Q_1$  are complex parameters with unit modulus we write the parameters as

$$Q_1 = e^{i\alpha} \quad q = e^{i\beta} \quad Q_2 = B e^{i\gamma} \tag{12}$$

and substitute these into equation (11) to obtain  $e^{i2\gamma} = e^{i(\alpha+2\beta)}$ . Substituting  $Q_2$  into the oscillator algebra we obtain

$$\begin{aligned} aa^* - Q_1 a^* a &= Bq Q_1^{1/2} q^{2N} + Bq^{-1} Q_1^{1/2} Rq^{-2N} \\ aa^* - Q_1^{-1} a^* a &= Bq^{-1} Q_1^{-1/2} q^{-2N} + Bq Q_1^{-1/2} Rq^{2N}. \end{aligned} \tag{13}$$

Then the spectrum turns out to be

$$a_n^* a_n = B \frac{\sin(2n\beta)}{\sin\left(\frac{2\beta-\alpha}{2}\right)} = B \frac{q^{2n} - q^{-2n}}{q Q_1^{-1/2} - q^{-1} Q_1^{1/2}}. \tag{14}$$

The positive definiteness of  $a_n^* a_n$  implies that the representation should be finite dimensional with the states  $|0\rangle \cdots |k-1\rangle$  and the highest state should be annihilated by the raising operator  $a^*|k-1\rangle = 0$ . This gives that  $q$  is a root of unity  $q = e^{i\frac{\pi}{2k}}$  and taking  $\frac{\pi}{k} - 2\pi < \alpha < \frac{\pi}{k}$  the spectrum turns out to be

$$a_n^* a_n = B \frac{\sin\left(n\frac{\pi}{k}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{k} - \alpha\right)}. \tag{15}$$

Although the deformation parameter  $Q_1$  (or  $\alpha$ ) seems only to be a free scaling parameter in the spectrum, its value will be fixed by the Hopf algebra structure.

### 3. Braided Hopf algebra

The generalization of the permutation map of boson/fermion algebras in tensor product space by a generalized map leads naturally to the generalization of the Hopf algebra. This is called

braided Hopf algebra whose axioms in algebraic (not diagrammatic) form are

$$\begin{aligned}
m \circ (\text{id} \otimes m) &= m \circ (m \otimes \text{id}) \\
m \circ (\text{id} \otimes \eta) &= m \circ (\eta \otimes \text{id}) = \text{id} \\
(\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
(\epsilon \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \epsilon) \circ \Delta = \text{id} \\
m \circ (\text{id} \otimes S) \circ \Delta &= m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon \\
\psi \circ (m \otimes \text{id}) &= (\text{id} \otimes m) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) \\
\psi \circ (\text{id} \otimes m) &= (m \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (\psi \otimes \text{id}) \\
(\text{id} \otimes \Delta) \circ \psi &= (\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (\Delta \otimes \text{id}) \\
(\Delta \otimes \text{id}) \circ \psi &= (\text{id} \otimes \psi)(\psi \otimes \text{id}) \circ (\text{id} \otimes \Delta) \\
\Delta \circ m &= (m \otimes m)(\text{id} \otimes \psi \otimes \text{id}) \circ (\Delta \otimes \Delta) \\
S \circ m &= m \circ \psi \circ (S \otimes S) \\
\Delta \circ S &= (S \otimes S) \circ \psi \circ \Delta \\
\epsilon \circ m &= \epsilon \otimes \epsilon \\
(\psi \otimes \text{id}) \circ (\text{id} \otimes \psi) \circ (\psi \otimes \text{id}) &= (\text{id} \otimes \psi) \circ (\psi \otimes \text{id}) \circ (\text{id} \otimes \psi)
\end{aligned} \tag{16}$$

where  $m : A \otimes A \rightarrow A$  is the multiplication map,  $\Delta : A \rightarrow A \otimes A$  is the comultiplication map,  $\eta : K \rightarrow A$  is the unit map,  $\epsilon : A \rightarrow K$  is the counit map,  $S : A \rightarrow A$  is the antipode map and  $\psi : A \otimes A \rightarrow A \otimes A$  is the braiding map. Note that in the limit  $\psi \rightarrow \pm\pi$  the braided Hopf algebra axioms reduce to the ordinary/super Hopf algebra axioms. The consistency of the braided Hopf algebra axioms (16) requires that the identity element  $1_A$  in any algebra should satisfy the following conditions:

$$\begin{aligned}
\Delta(1_A) &= 1_A \otimes 1_A & S(1_A) &= 1_A & \epsilon(1_A) &= 1 \\
\psi(1_A \otimes a) &= a \otimes 1_A & \psi(a \otimes 1_A) &= 1_A \otimes a & \forall a \in A.
\end{aligned} \tag{17}$$

The  $*$ -structure for the braided algebra  $B$  satisfies

$$\Delta \circ * = \pi \circ (* \otimes *) \circ \Delta \quad S \circ * = * \circ S. \tag{18}$$

If we interpret the Hopf algebra to define a system (of oscillators, for example) then the  $*$ -structure should be defined in such a way that it should be compatible with the algebra in the tensor product space. If  $a_1 \equiv a \otimes 1_A$  and  $b_2 \equiv 1_A \otimes b$  then

$$(a \otimes b)^* = ((a \otimes 1)(1 \otimes b))^* = (a_1 b_2)^* = b_2^* a_1^* = \psi(b^* \otimes a^*) \tag{19}$$

where in the non-braided limit it gives the ordinary  $*$ -structure  $(a \otimes b)^* = a^* \otimes b^*$ . We also note that the  $*$ -involution we define by equation (19) for the tensor product is different from the  $*$ -involution defined by Majid [5].

To find the Hopf algebra structure of the oscillator algebra (3) we start with the general forms of the coproducts and braidings which are linear in both factors of the tensor product. Although the nonlinear factors are also possible theoretically, we do not consider them in this paper. Thus, we start with the most general linear forms of the coproducts, braidings and antipodes, e.g.

$$\Delta(q^N) = D_1 q^N \otimes q^N + F_1 q^N \otimes q^{-N} + G_1 a \otimes a^* + H_1 a^* \otimes a + K_1 1_A \otimes q^N \dots \tag{20}$$

$$\psi(q^N \otimes q^N) = g_1 q^N \otimes q^N + g_2 a \otimes a^* + g_3 a^* \otimes a + g_4 1_A \otimes q^N + g_6 1_A \otimes 1_A + \dots \tag{21}$$

and similar expressions for the other generators of the algebra. It seems that it is very hard to solve the equations for the braided Hopf algebra because there are too many parameters. When

we substitute the general forms into the relation  $q^N q^{-N} = q^{-N} q^N = 1_A$  and use equation (17) we obtain that the braidings of  $q^N$  and  $q^{-N}$  with other generators are trivial, i.e. they are bosonic elements. From these relations we also obtain that

$$\begin{aligned} \Delta(q^N) &= D_1 q^N \otimes q^N & \Delta(q^{-N}) &= D_1^{-1} q^{-N} \otimes q^{-N} \\ S(q^N) &= S_1 q^{-N} & S(q^{-N}) &= S_1^{-1} q^N \end{aligned} \tag{22}$$

where  $D_1$  and  $S_1$  are complex parameters. This simple form (22) greatly simplifies the solution. Using other relations of the algebra, we obtain that the coproducts, the braidings and the antipodes for  $a$  and  $a^*$  are in the form

$$\begin{aligned} \Delta(a) &= D_2 a \otimes q^N + D_3 q^{-N} \otimes a & \Delta(a^*) &= D_3^* a^* \otimes q^N + D_2^* q^{-N} \otimes a^* \\ S(a) &= -S_2 a & S(a^*) &= -S_2^* a^* \\ \psi(a \otimes a) &= z a \otimes a & \psi(a^* \otimes a^*) &= z a^* \otimes a^* \\ \psi(a \otimes a^*) &= z^{-1} a^* \otimes a & \psi(a^* \otimes a) &= z^{-1} a \otimes a^* \end{aligned} \tag{23}$$

where  $D_2, D_3$  and  $S_2$  are complex parameters and  $z$  is a complex parameter of unit modulus.

We substitute these forms into the braided Hopf algebra axioms (16) and we find that there are only two solutions: one is the bosonic generalization of the exclusion with the exchange phase  $z = Q_1 = 1$ , and the other is the fermionic generalization of the exclusion with the exchange phase  $z = Q_1 = -1$ .

### 3.1. Generalized exclusion of the bosonic type

For the first solution we obtain  $Q_1 = 1$ , and all braidings turn out to be trivial ( $z = 1$ ). The oscillator algebra

$$\begin{aligned} aa^* - a^*a &= Q_2(q^{2N} + q^{-2(N+1)}) & aq^N &= qa^N a \\ q^N a^* &= qa^* q^N & aq^{-N} &= q^{-1} q^{-N} a \\ q^{-N} a^* &= q^{-1} a^* q^{-N} & q^N q^{-N} &= q^{-N} q^N = 1_A & a^k &= (a^*)^k = 0 \end{aligned} \tag{24}$$

has the Hopf algebra structure

$$\begin{aligned} \Delta(q^N) &= D_1 q^N \otimes q^N & \Delta(q^{-N}) &= D_1^{-1} q^{-N} \otimes q^{-N} \\ \Delta(a) &= D_2 a \otimes q^N + D_1^{-2} D_2 q^{-N} \otimes a & \Delta(a^*) &= D_1^2 D_2^* a^* \otimes q^N + D_2^* q^{-N} \otimes a^* \\ S(q^N) &= D_1^{-2} q^{-N} & S(q^{-N}) &= D_1^2 q^N \\ S(a) &= -q^{-1} a & S(a^*) &= -qa^* \\ \epsilon(q^N) &= D_1^{-1} & \epsilon(q^{-N}) &= D_1 \\ \epsilon(a) &= \epsilon(a^*) = 0 \end{aligned} \tag{25}$$

where  $q = e^{i\frac{\pi}{2k}}$ ,  $D_1^4 = -q^2$ ,  $D_2^* D_2 = 1$  and  $Q_2 = qB$ , where  $B$  is a free positive parameter. Since all the braidings turn out to be trivial ( $\psi = \pi$ ) the coproducts, antipodes and counits satisfy the ordinary Hopf algebra axioms. Choosing  $B = \frac{1}{2}$  the relation between  $a$  and  $a^*$  can equivalently be written as

$$aa^* - a^*a = \cos \frac{(2N+1)\pi}{2k} \tag{26}$$

and the spectrum is given by

$$a^* a |n\rangle = \frac{\sin \frac{n\pi}{k}}{2 \sin \frac{\pi}{2k}} |n\rangle. \tag{27}$$

This oscillator algebra, in the  $k \rightarrow \infty$  limit, reduces to the bosonic oscillator and hence we call it the generalized exclusion algebra of the bosonic type. We note that the exchange phase is constant (+1) and independent of  $k$ .

3.2. Generalized exclusion of the fermionic type

For the second solution we obtain  $Q_1 = -1$ , and the exchange phase for the creation and annihilation operators turn out to be  $-1$  ( $z = -1$ ). The oscillator algebra

$$\begin{aligned} aa^* + a^*a &= Q_2(q^{2N} - q^{-2(N+1)}) & aq^N &= qq^Na \\ q^Na^* &= qa^*q^N & aq^{-N} &= q^{-1}q^{-N}a \\ q^{-N}a^* &= q^{-1}a^*q^{-N} & q^Nq^{-N} &= q^{-N}q^N = 1_A & a^k &= (a^*)^k = 0 \end{aligned} \tag{28}$$

has the Hopf algebra structure

$$\begin{aligned} \Delta(q^N) &= D_1q^N \otimes q^N & \Delta(q^{-N}) &= D_1^{-1}q^{-N} \otimes q^{-N} \\ \Delta(a) &= D_2a \otimes q^N + D_1^{-2}D_2q^{-N} \otimes a & \Delta(a^*) &= D_1^2D_2^*a^* \otimes q^N + D_2^*q^{-N} \otimes a^* \\ S(q^N) &= q^{-1}q^{-N} & S(q^{-N}) &= qq^N \\ S(a) &= -q^{-1}a & S(a^*) &= -qa^* \\ \epsilon(q^N) &= D_1^{-1} & \epsilon(q^{-N}) &= D_1 & \epsilon(a) &= \epsilon(a^*) = 0 \end{aligned} \tag{29}$$

where  $D_1^4 = q^2$ ,  $D_2^*D_2 = 1$ ,  $q = e^{i\frac{\pi}{2k}}$  and  $Q_2 = Bq^{1-k} = -Bq^{k+1} = -iqB$ . The braiding relations are

$$\begin{aligned} \psi(a \otimes a) &= -a \otimes a & \psi(a^* \otimes a^*) &= -a^* \otimes a^* \\ \psi(a \otimes a^*) &= -a^* \otimes a & \psi(a^* \otimes a) &= -a \otimes a^*. \end{aligned} \tag{30}$$

Identifying

$$a_1 \equiv a \otimes 1_A \quad a_2 \equiv 1_A \otimes a \quad a_1^* \equiv a^* \otimes 1_A \quad a_2^* \equiv 1_A \otimes a^* \tag{31}$$

the braiding relations can equivalently be written as

$$a_2a_1 = -a_1a_2 \quad a_2^*a_1^* = -a_1^*a_2^* \quad a_2a_1^* = -a_1^*a_2 \quad a_2^*a_1 = -a_1a_2^*. \tag{32}$$

The relation between  $a$  and  $a^*$  can equivalently be written as

$$aa^* + a^*a = 2B \sin \frac{(2N+1)\pi}{2k} \tag{33}$$

and the spectrum is given by

$$a^*a|n\rangle = B \frac{\sin \frac{n\pi}{k}}{\cos \frac{\pi}{2k}} |n\rangle. \tag{34}$$

Choosing  $B = \frac{1}{\sqrt{2}}$  this oscillator gives the usual fermion algebra for  $k = 2$  and hence we call it the generalized exclusion algebra of the fermionic type. We note that the exchange phase is constant ( $-1$ ) and independent of  $k$ .

4. Covariance

To study the covariance properties we start with the generalized oscillator algebra (3) and make the transformation

$$\begin{pmatrix} a \\ a^* \\ q^N \\ q^{-N} \end{pmatrix}' = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12}^* & t_{11}^* & t_{14}^* & t_{13}^* \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{32}^* & t_{31}^* & t_{34}^* & t_{33}^* \end{pmatrix} \begin{pmatrix} a \\ a^* \\ q^N \\ q^{-N} \end{pmatrix}. \tag{35}$$

The covariance of  $(q^Nq^{-N} = q^{-N}q^N = 1_A)'$  gives

$$t_{31} = t_{32} = t_{34} = 0 \quad t_{33}^*t_{33} = 1 \tag{36}$$

and the covariance of  $(aa^* - Q_1a^*a = Q_2q^{2N} + Q_3q^{-2N})'$  gives

$$\begin{aligned}
 t_{11}t_{12}^* - Q_1t_{12}^*t_{11} &= 0 \\
 t_{12}t_{11}^* - Q_1t_{11}^*t_{12} &= 0 \\
 Q_2t_{11}t_{11}^* - Q_1Q_2t_{12}^*t_{12} + t_{13}t_{14}^* - Q_1t_{14}^*t_{13} &= Q_2t_{33}^2 \\
 Q_3t_{11}t_{11}^* - Q_1Q_3t_{12}^*t_{12} + t_{14}t_{13}^* - Q_1t_{13}^*t_{14} &= Q_3(t_{33}^*)^2 \\
 t_{11}t_{11}^* - t_{11}^*t_{11} + Q_1^{-1}t_{12}t_{12}^* - Q_1t_{12}^*t_{12} &= 0 \\
 qt_{13}t_{11}^* - Q_1t_{11}^*t_{13} + t_{12}t_{14}^* - qQ_1t_{14}^*t_{12} &= 0 \\
 q^{-1}t_{11}t_{13}^* - Q_1t_{13}^*t_{11} + t_{14}t_{12}^* - q^{-1}Q_1t_{12}^*t_{14} &= 0 \\
 qt_{11}t_{14}^* - Q_1t_{14}^*t_{11} + t_{13}t_{12}^* - qQ_1t_{12}^*t_{13} &= 0 \\
 q^{-1}t_{14}t_{11}^* - Q_1t_{11}^*t_{14} + t_{12}t_{13}^* - q^{-1}Q_1t_{13}^*t_{12} &= 0 \\
 t_{13}t_{13}^* + t_{14}t_{14}^* - Q_1t_{13}^*t_{13} - Q_1t_{14}^*t_{14} &= 0.
 \end{aligned} \tag{37}$$

The compatibility of these relations with the  $*$ -operation gives  $Q_1 = \pm 1$  and  $Q_3 = Q_2^*$ . For  $Q_1 = q = 1$  the entries of the transformation matrix commute among themselves, and setting  $t_{33} = 1$  and  $t_{13} = t_{14} = 0$  this gives the group  $SU(1, 1)$ . For  $q$  a root of unity, the covariance of the nilpotency condition  $(a')^k = 0$  gives  $t_{12} = 0$ . For  $Q_1 = -1$  the final equation in (37) requires  $t_{13} = t_{14} = 0$  and the transformation for the generalized exclusion of the fermionic type given by equation (28) turns out to be trivial. The algebra for the generalized exclusion of the bosonic type given by (24) is covariant under the transformation

$$\begin{pmatrix} a \\ a^* \\ q^N \\ q^{-N} \end{pmatrix}' = \begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & t_{11}^* & t_{14}^* & t_{13}^* \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{33}^* \end{pmatrix} \begin{pmatrix} a \\ a^* \\ q^N \\ q^{-N} \end{pmatrix} \tag{38}$$

if the entries of the transformation matrix satisfy

$$\begin{aligned}
 t_{11}t_{11}^* &= t_{11}^*t_{11} & t_{11}t_{13} &= q^3t_{13}t_{11} & t_{11}t_{13}^* &= qt_{13}^*t_{11} & t_{11}t_{14} &= q^{-3}t_{14}t_{11} \\
 t_{11}t_{14}^* &= q^{-1}t_{14}^*t_{11} & t_{11}t_{33} &= t_{33}t_{11} & t_{11}t_{33}^* &= t_{33}^*t_{11} & t_{13}t_{14} &= q^{-4}t_{14}t_{13} \\
 t_{13}t_{14}^* - t_{14}^*t_{13} + Q_2t_{11}t_{11}^* &= Q_2t_{33}^2 & t_{13}t_{13}^* - t_{14}^*t_{14} &= t_{13}^*t_{13} - t_{14}t_{14}^* \\
 t_{13}t_{33} &= qt_{33}t_{13} & t_{13}t_{33}^* &= q^{-1}t_{33}^*t_{13} & t_{14}t_{33} &= qt_{33}t_{14} \\
 t_{14}t_{33}^* &= q^{-1}t_{33}^*t_{14} & t_{33}^*t_{33} &= 1 & t_{13}^k &= t_{14}^k = 0
 \end{aligned} \tag{39}$$

and  $*$ -conjugates. The deformation parameter  $q$  is a root of unity,  $q = e^{i\frac{\pi}{2k}}$ . The coproducts

$$\begin{aligned}
 \Delta(t_{11}) &= t_{11} \otimes t_{11} & \Delta(t_{13}) &= t_{11} \otimes t_{13} + t_{13} \otimes t_{33} \\
 \Delta(t_{33}) &= t_{33} \otimes t_{33} & \Delta(t_{14}) &= t_{11} \otimes t_{14} + t_{14} \otimes t_{33}^* \\
 \Delta(t_{11}^*) &= t_{11}^* \otimes t_{11}^* & \Delta(t_{13}^*) &= t_{11}^* \otimes t_{13}^* + t_{13}^* \otimes t_{33}^* \\
 \Delta(t_{33}^*) &= t_{33}^* \otimes t_{33}^* & \Delta(t_{14}^*) &= t_{11}^* \otimes t_{14}^* + t_{14}^* \otimes t_{33}
 \end{aligned} \tag{40}$$

the antipodes

$$\begin{aligned}
 S(t_{11}) &= t_{11}^{-1} & S(t_{13}) &= -t_{11}^{-1}t_{13}t_{33}^{-1} & S(t_{14}) &= -t_{11}^{-1}t_{14}t_{33} \\
 S(t_{33}) &= t_{33}^{-1} & S(t_{11}^*) &= (t_{11}^*)^{-1} & S(t_{13}^*) &= -(t_{11}^*)^{-1}t_{13}^*t_{33} \\
 S(t_{14}^*) &= -(t_{11}^*)^{-1}t_{14}^*t_{33}^{-1} & S(t_{33}^*) &= t_{33}
 \end{aligned} \tag{41}$$

and counits

$$\epsilon(t_{11}) = \epsilon(t_{11}^*) = \epsilon(t_{33}) = \epsilon(t_{33}^*) = 1 \quad \epsilon(t_{13}) = \epsilon(t_{13}^*) = \epsilon(t_{14}) = \epsilon(t_{14}^*) = 0 \tag{42}$$

complete the Hopf algebra structure.



## 5. Conclusion

In this paper we have investigated the braided Hopf algebra structure of the generalized oscillator with generalized exclusion  $a^k = 0$ , i.e., there can be at most  $k - 1$  particles at the same state. We find that there are two solutions. One is the generalized exclusion of the bosonic type where the exchange phase is  $(+1)$  and the limit  $k \rightarrow \infty$  gives the bosonic oscillator. The second solution is the generalized exclusion of the fermionic type where the exchange phase is  $(-1)$  and the limit  $k \rightarrow 2$  gives the fermionic oscillator. We note that in the operator algebra with exclusion  $a^k = 0$  we start with the generalized exchange phases (braidings) but the Hopf algebra structure chooses only the  $\pm 1$  factor. If we do not impose any exclusion on the creation–annihilation operators it is possible to find Hopf algebra solutions with the generalized exchange phase (braidings) [4]. Hence, in the context of Hopf algebras, we arrive at the conclusion that the generalization of exclusion does not necessarily imply the generalization of the exchange phase and vice versa.

The Hopf algebras we have found for the oscillators can be used to generate the Hopf structure of Lie algebras through a realization using some extension of the Jordan–Schwinger map, just like those discussed for parabosonic and parafermionic algebras [14].

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